Problem Set on Differential Equations

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Problem 1

Find the solution of the initial value problem

$$\dot{a}(t) = r \cdot a(t) + s$$
$$a(0) = a_0$$

where both r and s are known constant.

Problem 2

Find the solution of the initial value problem

$$\dot{a}(t) = r(t) \cdot a(t) + s(t)$$
$$a(0) = a_0$$

where both $r(t)$ and $s(t)$ are known functions of t.

Problem 3

Consider the linear system of differential equations given by

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \boldsymbol{x}(t).$$

A) Find the general solution of the system.

B) What would you need to find a specific solution of the system?

C) Draw the trajectories of the system.

Problem 4

Consider the initial value problem

$$\dot{k}(t) = s \cdot f\,(k(t)) - \delta \cdot k(t)$$
$$k\,(0) = k_0$$

where the saving rate $s \in (0,1)$, the capital depreciation rate $\delta \in (0,1)$, and the production function f satisfies the *Inada conditions*. That is, f is continuously differentiable and

$$f(0) = 0$$
$$f'(x) > 0$$
$$f''(x) < 0$$
$$\lim_{x \to 0} f'(x) = +\infty$$
$$\lim_{x \to +\infty} f'(x) = 0.$$

A) Give a production function f that satisfies the Inada conditions.

B) Find the steady state of the system.

C) Draw the dynamic path of $k(t)$ and show that it converges to the steady state.

Problem 5

The solution of the problem studied in Problem 4 is characterized by a system of two nonlinear first-order differential equations:

$$\dot{k}_t = f\left(k_t\right) - c_t - \delta \cdot k_t$$
$$\frac{\dot{c}_t}{c_t} = \alpha \cdot A \cdot k_t^{\alpha-1} - (\delta + \rho)\ .$$

The first differential equation is the law of motion of capital. The second differential equation is the Euler equation, which describes the optimal path of consumption over time.

A) Draw the phase diagram of the system.

B) Linearize the system around its steady state.

C) Show that the steady state is a saddle point locally.

D) Suppose the economy is in steady state at time t_0 and there is an unanticipated decrease in the discount factor ρ. Show on your phase diagram the transition dynamics of the model.

Problem 6

The solution of the investment problem studied in Problem 5 is characterized by a system of two nonlinear first-order differential equations:

$$\dot{k}_t = \left(\frac{q_t - 1}{\chi}\right) \cdot k_t$$
$$\dot{q}_t = r \cdot q_t - f'(k_t) - \frac{1}{2 \cdot \chi}(q_t - 1)^2 .$$

The first differential equation is the law of motion of capital k_t. The second differential equation is the law of motion of the co-state variable q_t.

A) Draw the phase diagram.

B) Show that the steady state is a saddle point locally.

Problem 7

Consider a discrete time version of the typical growth model:

$$k(t+1) = f(k(t)) - c(t) + (1-\delta) \cdot k(t)$$
$$c(t+1) = \beta \cdot \left[1 + f'(k(t)) - \delta\right] \cdot c(t).$$

The discount factor $\beta \in (0,1)$, the rate of depreciation of capital $\delta \in (0,1)$, initial capital k_0 is given, and the production function f satisfies the Inada conditions. These two equations are a system of first-order difference equations. Whereas a system of first-order differential equations relates $\dot{\boldsymbol{x}}(t)$ to $\boldsymbol{x}(t)$, a system of first-order difference equations relate $\boldsymbol{x}(t+1)$ to $\boldsymbol{x}(t)$.

We will see that we can study a system of first-order difference equations with the tools that we used to study systems of first-order differential equations. In particular, we can use phase diagrams to understand the dynamics of the system.

A) Construct a phase diagram for the system. First, define

$$\Delta k \equiv k(t+1) - k(t),$$
$$\Delta c \equiv c(t+1) - c(t).$$

Second, draw the $\Delta k = 0$ locus and the $\Delta c = 0$ locus on the (k, c) plane. Finally, find the steady state as the intersection of the $\Delta k = 0$ locus and the $\Delta c = 0$ locus.

B) Show that the steady state is a saddle point in the phase diagram.

Problem 8

We consider the following optimal growth problem. Given initial human capital h_0 and initial physical capital k_0, choose consumption $c(t)$ and labor $l(t)$ to maximize utility

$$\int_0^{\infty} e^{-\rho \cdot t} \cdot \ln(c) dt$$

subject to

$$\dot{k}_t = y_t - c_t - \delta \cdot k_t$$
$$\dot{h}_t = B \cdot (1 - l_t) \cdot h_t.$$

Output y_t is defined by

$$y_t \equiv A \cdot k_t^{\alpha} \cdot (l_t \cdot h_t)^{\beta}.$$

We also impose that $0 \leq l_t \leq 1$. The discount factor $\rho > 0$, the rate of depreciation of physical capital $\delta > 0$, the constants $A > 0$ and $B > 0$, and the production function parameters $\alpha \in (0, 1)$ and $\beta \in (0, 1)$.

A) Give state and control variables.

B) Write down the present-value Hamiltonian for this problem.

C) Derive the optimality conditions.

D) Show that the growth rate of consumption $c(t)$ is

$$\frac{\dot{c}}{c} = \frac{\alpha \cdot y}{k} - (\delta + \rho).$$

E) From now on, we assume that $B = 0$. Show that $l = 1$.

F) Draw the phase diagram in the (k, c) plane.

G) Show on the diagram that the steady state of the system is a saddle point.

H) Derive the Jacobian of the system.

I) Show that the steady state of the system is a saddle point.